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Families of Liapunov Functions for Nonlinear Systems in Critical Cases

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Families of Liapunov Functions for Nonlinear Systems in Critical Cases

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Abstract

Liapunov functions are constructed for nonlinear systems of ordinary differential equations whose linearized system at an equilibrium point possesses either a simple zero eigenvalue or a complex conjugate pair of simple, pure imaginary eigenvalues. The construction is explicit, and yields parametrized families of Liapunov functions for such systems. In the case of a zero eigenvalue, the Liapunov functions contain quadratic and cubic terms in the state. Quartic terms appear as well for the case of a pair of pure imaginary eigenvalues. Predictions of local asymptotic stability using these Liapunov functions are shown to coincide with those of pertinent bifurcation-theoretic calculations. The development of the paper is carried out using elementary properties of multilinear functions. The Liapunov function families thus obtained are amenable to symbolic computer coding.

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1 INTRODUCTION

In this paper, we construct families of Liapunov functions useful in assessing the asymptotic stability of critical equilibrium points of a class of systems

$$\dot{x} = f(x) \tag{1}$$

where $x \in I\!R^n$ and f is at least four times continuously differentiable. Throughout the presentation, we let the origin $x = 0$ be the equilibrium point of interest of this system of ordinary differential equations. *Critical cases* in the study of stability of the origin of Eq. (1) are those in which the Jacobian matrix $Df(0)$ possesses at least one eigenvalue with zero real part, and no eigenvalues with positive real part. In these situations, it is not possible to ascertain whether or not the origin of (1) is locally asymptotically stable solely from the linearized system $\dot{x} = (Df(0))x$.

We focus on two specific critical cases in the stability analysis of Eq. (1). These correspond to the Jacobian matrix of (1) at the origin possessing either a simple zero eigenvalue or a complex conjugate pair of simple, pure imaginary eigenvalues. The associated hypotheses ((S) and (H) below, respectively) are introduced next. The techniques of this paper can also be applied to other critical cases, such as those involving multiple critical eigenvalues [7].

In essence, the paper relies on two basic tools in achieving its goals. First, the notation of multilinear functions is adhered to throughout the paper, in denoting terms in the Taylor series expansions of the nonlinear system of interest as well as Liapunov function candidates and their derivatives. Second, a new result on local definiteness of a class of scalar bivariate functions is introduced. This appears as Lemma 1 in Section 4. Lemma 1 is indeed key to the subsequent constructions of Liapunov functions in Cases (S) and (H). These constructions are rather mechanical given Lemma 1, albeit somewhat tedious.

The first critical case of interest in this work is characterized by the occurrence of a simple zero eigenvalue of $Df(0)$, with all remaining eigenvalues having strictly negative real parts. As discussed in [2], stability of the origin in this situation is closely related to the stability of bifurcated equilibrium points in smooth parametrizations of Eq. (1). Because of this connection to stationary (or static) bifurcation, this critical case will be referred to here as "Case (S)."

(S) The Jacobian $Df(0)$ possesses a simple zero eigenvalue, with all other eigenvalues in the open left half of the complex plane.

In the second critical case of interest here, $Df(0)$ is assumed to possess a single complex conjugate pair of simple, pure imaginary eigenvalues, with the remaining eigenvalues lying in the open left half complex plane. Reference [1] discusses the relationship of stability of the origin in this case with stability of bifurcated periodic solutions of smooth parametrizations of (1). Hopf bifurcation to periodic solutions occurs for generic such parametrizations of Eq. (1) under these circumstances. Because of this connection to Hopf bifurcation, this critical case will be referred to as "Case (H)."

(H) The Jacobian $Df(0)$ possesses a complex conjugate pair of simple, pure imaginary eigenvalues, with all other eigenvalues in the open left half of the complex plane.

Recently, there has been significant interest in feedback stabilization of nonlinear systems in critical cases (see, e.g., the review paper [18] and references therein). This has yielded various existence and synthesis results on stabilizability by either smooth or continuous feedback. The main contribution of the present paper to this body of work is the construction of new Liapunov functions for such systems, in the two critical cases (S) and (H). Liapunov functions facilitate estimation of the domain of attraction of a stable equilibrium point, and as such can serve to quantify the efficacy of a given control design. Liapunov functions can be used to define performance indices in optimization-based feedback control design of nonlinear systems. Such performance indices can involve estimates of the achieved domain of attraction and measures of the adequacy of the transient response. The Liapunov functions derived here for Cases (S) and (H) are given explicitly in terms of the system's dynamics, and are amenable to symbolic computer coding. As a by-product of our results, known formulae for testing stability in the critical cases (S) and (H) (so-called *bifurcation formulae*) are found to follow easily from the Liapunov functions we obtain. Note, however, that these formulae alone do not yield *analytical* performance indices of the type just alluded to.

The asymptotic stability of nonlinear systems in critical cases has received significant attention in the literature (e.g., [3], [11], [13]-[15], [17], [19]). Although some of these works have employed Liapunov stability analysis, the Liapunov functions used have generally been defined only *implicitly*. In some cases, this is linked to the use of implicitly defined nonlinear coordinate transformations to lower dimensional problems. Implicitly defined Liapunov functions suffice when the goal of the analysis is limited to deriving sufficient conditions for local asymptotic stability. For instance, one result of Mees and Chua [17] gives a Liapunov function for *planar* systems (1) satisfying (H). Implications of this result for higher dimensional models (1) follow from the Center Manifold

Theorem (cf. [17],[5]). In the present paper, we give an *explicit* construction of families of Liapunov functions for critical nonlinear systems satisfying either hypothesis (S) or (H) which apply *directly* to the given n-dimensional system description (1).

This paper is organized as follows. In Section 2, pertinent results on multilinear functions are given. The set-up for construction of Liapunov functions for systems (1) is formulated using multilinear function notation in Section 3. A lemma giving sufficient conditions for local definiteness of a class of scalar bivariate functions is presented in Section 4. Section 5 contains a result on solutions of Liapunov matrix equations for a coefficient matrix with a zero eigenvalue or a pair of pure imaginary eigenvalues. The main results of the paper appear in Sections 6 and 7. Section 6 contains an explicit construction of a family of Liapunov functions for Case S, and Section 7 contains an analogous construction for Case (H). Conclusions are collected in Section 8.

Notation. In what follows, $I\!R^n$ denotes the space of n-dimensional column vectors having real entries, while $\mathbb{C}^n$ denotes the space of n-dimensional column vectors with complex entries. The complex conjugate of a quantity (scalar, vector, or matrix) a is denoted by $\bar{a}$. The transpose of a vector or matrix a is denoted a^T. For a vector space V, denote by $(V)^k$ the vector space obtained as the k-tuple product $V \times \cdots \times V$. The Jacobian derivative of a function ϕ is denoted $D\phi$. The norm of a vector $x \in I\!R^n$ will be denoted $|x|$, and the same notation will apply to any compatible matrix norm. Denote by r (resp. l) the right column (resp. left row) eigenvector of $Df(0)$ corresponding to the critical eigenvalue 0 (Case (S)) or $i\omega_c$ (Case (H)). For consistency with previous literature [1], [2], [6], [9], the first component of r is set to unity, and l is then chosen subject to the normalization $lr = 1$. (Ensuring that the first component of r is nonzero in some cases requires a reordering of the elements of x.)

2 RESULTS ON MULTILINEAR FUNCTIONS

Multivariable Taylor series can be conveniently represented in terms of multilinear functions. We shall employ multilinear functions in representing Taylor series expansions both for the vector field $f(x)$ of Eq. (1), and for the Liapunov functions whose construction is the main purpose of this work. In this section, we present several useful facts pertaining to multilinear functions.

2.1 Multilinear Functions

Multilinear functions may be defined as follows.

Definition 1. Let V_1, V_2,...,V_k and W be vector spaces over the same field. A map $\psi : V_1 \times V_2 \times \ldots \times V_k \mapsto W$ is said to be *multilinear* (or *k-linear*) if it is linear in each of its variables. That is [4, p. 76], for arbitrary $v^i, \widetilde{v}^i \in V_i$, $i = 1, \ldots, k$, and for arbitrary scalars $a, \widetilde{a}$, we have

$$\psi(v^1, \ldots, av^i + \widetilde{a}\widetilde{v}^i, \ldots, v^k) = a\psi(v^1, \ldots, v^i, \ldots, v^k) + \widetilde{a}\psi(v^1, \ldots, \widetilde{v}^i, \ldots, v^k). \tag{2}$$

□

We refer to k as the *degree* of the multilinear function ψ. In particular, multilinear functions of degree two, three and four are referred to as *bilinear*, *trilinear* and *tetralinear functions*, respectively.

We shall in the sequel deal exclusively with multilinear functions ψ whose domain is the product space of k *identical* vector spaces $V_1 = V_2 = \ldots = V_k = V$. For such multilinear functions, we have the following notion of symmetry.

Definition 2. A k-linear function $\psi : V \times V \times \ldots \times V \mapsto W$ is *symmetric* if, for any $v^i \in V$, $i = 1, \ldots, k$, the vector

$$\psi(v^1, v^2, \ldots, v^k) \tag{3}$$

is invariant under arbitrary permutations of the argument vectors v^i. □

With an arbitrary multilinear function ψ, we associate a symmetric multilinear function ψ_s resulting from the following simple device, known as the *symmetrization operation* [4, pp. 88-89]. Given a multilinear function $\psi(x^1, x^2, \ldots, x^k)$, define a new (symmetric) multilinear function ψ_s as follows:

$$\psi_s(x^1, x^2, \ldots, x^k) := \frac{1}{k!} \sum_{(i_1, i_2, \ldots, i_k)} \psi(x^{i_1}, x^{i_2}, \ldots, x^{i_k}), \tag{4}$$

where the sum is taken over the $k!$ permutations of the integers $1, 2, ..., k$.

2.3 Coordinate Representation of Scalar Multilinear Functions

In this subsection, we state a useful representation result for scalar multilinear functions $\psi : (I\!R^n)^k \mapsto I\!R^m$. The representation rests upon a choice of basis ("coordinates") for $I\!R^n$. Thus, let $\{r^1, r^2, \ldots, r^n\}$ be a basis for $I\!R^n$. By a standard result [4, Proposition 3.6.1], to this basis there corresponds a unique *dual basis* which we may view as consisting of row vectors $l^1, l^2, \ldots, l^n$ such that

$$l^i r^j = \delta_{ij} \tag{9}$$

for $i, j = 1, \ldots, n$. Here, δ_{ij} is the Kronecker delta symbol:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{10}$$

for $i, j = 1, \ldots, n$. Eq. (9) will be referred to as the *biorthonormality property* of the vectors l^i, r^j.

Proposition 2 below provides a convenient representation for scalar symmetric multilinear functions on $(I\!R^n)^k$ in terms of the dual basis vectors l^i and a set of "structural coefficients." The result will be applied in the next section, yielding a representation of Liapunov function candidates.

Proposition 2. (Coordinate Representation of Scalar Multilinear Functions) Any symmetric k-linear function $\psi : (I\!R^n)^k \mapsto I\!R$ can be written as

$$\psi(x^1, x^2, \ldots, x^k) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \psi_{i_1 i_2 \ldots i_k} (l^{i_1} x^1)(l^{i_2} x^2) \ldots (l^{i_k} x^k) \tag{11}$$

where the (k-tuple) sum is taken over all $i_1, i_2, \ldots, i_k$, and where the structural coefficients $\psi_{i_1 i_2 \ldots i_k}$ are symmetric with respect to all permutations of $i_1, i_2, \ldots, i_k$.

Proof. This result is a special case of [16, Thm. 1.2]. However, we sketch a rather straightforward proof for the sake of completeness. By [4, Thm. 2.12.2], a scalar multilinear function is determined by its values when evaluated at all combinations of basis vectors as arguments. The formula (11) for ψ is clearly that of a k-linear function. Moreover, it follows from the biorthonormality property (9) of basis vectors r^i and dual basis vectors l^i that, by appropriate assignment of the structural coefficients $\psi_{i_1 i_2 \ldots i_k}$, any set of such values may be achieved. Hence, the representation above is sufficiently general to accomodate any scalar multilinear function ψ. □

2.4 Complexification of Real Multilinear Functions

It is sometimes convenient to evaluate a (real) multilinear function $\psi : (\mathbb{R}^n)^k \mapsto \mathbb{R}^m$ for argument vectors in $\mathbb{C}^n$. This is done simply by evaluating the value of ψ as if the argument vectors were in $\mathbb{R}^n$, using a representation of ψ such as Eq. (11) above. This process is the *complexification* of ψ.

Our use of the complexification device is relegated to Section 7, in the construction of Liapunov functions for Eq. (1) under hypothesis (H). The following observation will be important in ensuring that the constructed Liapunov functions are indeed real-valued.

Proposition 3. **(Test for Realness of Multilinear Functions)** Let ψ denote a symmetric k-linear function $\psi : (\mathbb{C}^n)^k \mapsto \mathbb{C}^m$. The image of $(\mathbb{R}^n)^k$ of under the map ψ is $\mathbb{R}^m$ if and only if

$$\overline{\psi(x^1, x^2, \ldots, x^k)} = \psi(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^k) \tag{12}$$

for all vectors $x^1, x^2, \ldots, x^k \in \mathbb{C}^n$.

Proof. The "*if* " part is automatic. An induction proof is now sketched for the "*only if* " part. Let j denote the number of argument vectors $x^i \in \mathbb{C}^n$ that are not also in $\mathbb{R}^n$. That (12) holds when $j = 0$ is obvious. Also, if (12) holds for some $j = j_0 < k$, then it is a simple exercise to verify that it also holds for $j = j_0 + 1$. □

3 REPRESENTATION OF LIAPUNOV FUNCTION CANDIDATES

Eq. (1) may be rewritten, upon Taylor series expansion of $f(x)$, in the form

$$\begin{aligned} \dot{x} &= f(x) \\ &= Lx + Q(x,x) + C(x,x,x) + \cdots \end{aligned} \tag{13}$$

Here, $L := Df(0)$ and $Q(x,x)$, $C(x,x,x)$ are vector-valued quadratic and cubic forms, with the dots denoting higher order terms. Without loss of generality, assume that $Q(x,x)$ is induced by a symmetric bilinear function $Q(x^1, x^2)$ and, similarly, that $C(x,x,x)$ is induced by a symmetric trilinear function $C(x^1, x^2, x^3)$.

In Case (S) (one zero eigenvalue), we shall in the sequel seek Liapunov functions $\mathcal{V}(x)$ consisting

of the sum of a quadratic part and a cubic part, viz.

$$\mathcal{V}(x) = x^T \mathcal{P} x + \mathcal{K}(x, x, x). \tag{14}$$

It is natural to require $\mathcal{P}$ to be symmetric and positive definite. Similarly, the cubic form $\mathcal{K}(x, x, x)$ is induced by a symmetric trilinear function $\mathcal{K}(x^1, x^2, x^3)$. Since the term $\mathcal{K}(x, x, x)$ is dominated by the quadratic term $x^T \mathcal{P} x$, any such $\mathcal{V}(x)$ will indeed be locally positive definite.

In our study of Case (H) (two purely imaginary eigenvalues), we will include a quartic term $\mathcal{T}(x, x, x, x)$ in the candidate Liapunov function $\mathcal{V}(x)$ in addition to the quadratic and cubic terms presented in (2), viz.

$$\mathcal{V}(x) = x^T \mathcal{P} x + \mathcal{K}(x, x, x) + \mathcal{T}(x, x, x, x). \tag{15}$$

We of course ask that the quartic form $\mathcal{T}(x, x, x, x)$ be induced by a symmetric tetralinear function $\mathcal{T}(x^1, x^2, x^3, x^4)$. Certainly, the local positive definiteness of $\mathcal{V}(x)$ with $\mathcal{P} > 0$ remains preserved under inclusion of $\mathcal{T}(x, x, x, x)$ or terms of still higher order.

Next, we invoke Proposition 2 for the cases $k = 2, 3$ and 4, obtaining coordinate representations of the bilinear, trilinear, and tetralinear functions ${x^1}^T \mathcal{P} x^2$, $\mathcal{K}(x^1, x^2, x^3)$, and $\mathcal{T}(x^1, x^2, x^3, x^4)$, respectively.

Consider first the case $k = 2$. Since any real quadratic form ${x^1}^T \mathcal{P} x^2$ is determined by a real symmetric matrix $\mathcal{P}$, we can apply Proposition 2 to conclude that all such matrices $\mathcal{P}$ have the form

$$\mathcal{P} = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{ij} {l^i}^T l^j, \tag{16}$$

where $\pi_{ij} = \pi_{ji}$ are real coeffiecients.

The following representations for trilinear functions $\mathcal{K}$ and tetralinear functions $\mathcal{T}$ also follow from Proposition 2:

$$\begin{aligned} \mathcal{K}(x^1, x^2, x^3) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \kappa_{ijk} (l^i x^1)(l^j x^2)(l^k x^3), & (17) \\ \mathcal{T}(x^1, x^2, x^3, x^4) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{p=1}^{n} \tau_{ijkp} (l^i x^1)(l^j x^2)(l^k x^3)(l^p x^4), & (18) \end{aligned}$$

respectively. These multilinear functions are rendered symmetric by imposing the condition that the values of the structural coefficients (κ_{ijk}) and (τ_{ijkp}) do not depend on the order of the indices.

The simple representations above for the bilinear, trilinear and tetralinear functions appearing in the Liapunov function candidates $\mathcal{V}$ imply that *the construction of $\mathcal{V}$ is tantamount to specification of the structural coefficients* $\pi_{ij}, \kappa_{ijk}, \tau_{ijkp}$.

In the calculations to follow, the goal is to obtain sets of structural coefficients $\pi_{ij}, \kappa_{ijk}, \tau_{ijkp}$ which ensure the local negative definiteness of $\dot{\mathcal{V}}$, the time derivative of the Liapunov function candidate, along trajectories of (1). Of course, this will only be possible under assumptions on Eq. (1) which guarantee local asymptotic stability of the origin. In the next section, a result is presented on local definiteness of a class of bivariate functions. The sufficient conditions for local definiteness provided by this result will facilitate a systematic derivation of local Liapunov functions and conditions for local asymptotic stability.

4 LOCAL DEFINITENESS OF A CLASS OF BIVARIATE FUNCTIONS

We now introduce an interesting lemma which will prove to be an important tool in exhibiting conditions for local negative definiteness of the time derivative of a Liapunov function candidate.

Lemma 1. (Local Definiteness of a Class of Bivariate Functions) The scalar bivariate function

$$\begin{aligned}\delta(u,v) &= a_{20}u^2 + a_{04}v^4 \\ &\quad + a_{21}u^2v + a_{30}u^3 + a_{13}uv^3 + a_{22}u^2v^2 + a_{31}u^3v + a_{40}u^4 + O(|(u,v)|^5) \qquad (19)\end{aligned}$$

in the real variables u and v is locally negative definite near $(u,v) = (0,0)$ provided that $a_{20} < 0$ and $a_{04} < 0$. Here, $O(|(u,v)|^5)$ denotes terms of fifth and higher order in $|(u,v)|$.

Remark 1. It is convenient to view this lemma as follows. Consider $\delta(u,v)$, a scalar polynomial function of the scalars u and v, for which the leading term in $\delta(u,0)$ is $a_{20}u^2$, and the leading term in $\delta(0,v)$ is $a_{04}v^4$. Then the assertion is that $\delta(u,v)$ is locally negative definite when two basic conditions are fulfilled: First, the *absence* of the two terms $a_{03}v^3$, $a_{12}uv^2$; Second, the local negative definiteness of the *uni*variate functions $\delta(u,0)$ and $\delta(0,v)$ (i.e., $a_{20} < 0$ and $a_{04} < 0$, respectively).

Proof. In proving Lemma 1, we neglect the terms $O(|(u,v)|^5)$ in $\delta(u,v)$, since, being higher order terms, they can easily be incorporated with only slight modifications in the analysis. With this

understanding, rewrite $\delta(u,v)$ in the form of a quadratic polynomial in u:

$$\begin{aligned}\delta(u,v) &= (a_{20}+a_{21}v+a_{30}u+a_{22}v^2+a_{31}uv+a_{40}u^2)u^2+a_{13}v^3u+a_{04}v^4 \\ &=: p(u,v)u^2+(a_{13}v^3)u+a_{04}v^4. \end{aligned} \tag{20}$$

Here,

$$p(u,v) := a_{20}+a_{21}v+a_{30}u+a_{22}v^2+a_{31}uv+a_{40}u^2. \tag{21}$$

Since $p(0,0)=a_{20}<0$, it is clear that there is an $\epsilon_1>0$ such that $p(u,v)<0$ for

$$|u|,|v|<\epsilon_1. \tag{22}$$

(One could easily write a formula for such an ϵ_1.)

The leading coefficient $p(u,v)$ in the expression (20) for $\delta(u,v)$ is therefore strictly negative for $|u|,|v|<\epsilon_1$. Next, rewrite $\delta(u,v)$ as

$$\delta(u,v)=p(u,v)[u+\frac{a_{13}v^3}{2p(u,v)}]^2+q(u,v), \tag{23}$$

where

$$\begin{aligned} q(u,v) := \frac{v^4}{4p(u,v)}\{4a_{20}a_{04} &+ 4a_{30}a_{04}u+4a_{21}a_{04}v \\ &+ 4a_{04}a_{40}u^2+4a_{04}a_{31}uv+(4a_{04}a_{22}-a_{13}^2)v^2\}. \end{aligned} \tag{24}$$

Since $a_{20}<0$ and $a_{40}<0$, the constant term in the expression in braces in Eq. (24), namely $4a_{20}a_{04}$, is strictly positive. Hence, there is an $\epsilon_2>0$ such that the expression in braces in Eq. (24) is strictly positive for

$$|u|,|v|<\epsilon_2. \tag{25}$$

Recalling that $p(u,v)<0$ for $|u|,|v|<\epsilon_1$, we have that for $|u|,|v|<\epsilon:=\min(\epsilon_1,\epsilon_2)$, $q(u,v)\le 0$, with $q(u,v)=0$ *only for* $v=0$ (see Eq. (24)). Now consider the implications of these observations for the expression (23) for $\delta(u,v)$. Clearly, for $|u|,|v|<\epsilon$ and $v\neq 0$, the fact that $q(u,v)$ is strictly negative ensures that $\delta(u,v)<0$. If, on the other hand, $|u|,|v|<\epsilon$ and $v=0$, then $\delta(u,v)$ reduces to

$$\begin{aligned}\delta(u,0) &= p(u,0)u^2 \\ &< 0\end{aligned} \tag{26}$$

for $u\neq 0$. Thus, $\delta(u,v)$ is indeed locally negative definite near $(0,0)$. □

5 CALCULATIONS INVOLVING THE STABLE SUBSPACE

In this section, we define the stable subspace of $I\!R^n$ corresponding to the Jacobian matrix L, recall an associated orthogonality property from [10], and employ the stable subspace concept in the choice of the quadratic term $x^T \mathcal{P} x$ in the Liapunov function candidate $\mathcal{V}(x)$ (cf. Eqs. (14), (15)). The development proceeds for Cases (S) and (H) in parallel.

Definition 4. The *stable subspace* of $I\!R^n$, denoted by E^s, is the span of the eigenvectors (and generalized eigenvectors, if any) of L corresponding to the stable eigenvalues of L. □

In Case S, any vector $x \in I\!R^n$ has a unique representation $x = ar + w$ where a is a real scalar, r is the right eigenvector of L corresponding to the eigenvalue 0, and $w \in E^s$. In Case H, any vector $x \in I\!R^n$ has a unique representation $x = ar + \bar{a}\bar{r} + w$ where a is a complex scalar, r is the right eigenvector of L corresponding to the eigenvalue $i\omega_c$, and $w \in E^s$.

The following property is well known (see, e.g., [10, Appendix 4.1]).

Proposition 4. (Orthogonality of Left and Right Eigenvectors) Let l^α and r^β denote left and right eigenvectors, respectively, corresponding to eigenvalues λ_α and λ_β of a matrix A. Either, or both, of l^α and r^β may be generalized eigenvectors. If $\lambda_\alpha \neq \lambda_\beta$, then $l^\alpha r^\beta = 0$. Moreover, the subspace of all column vectors nullified by l^α is precisely the span of all right eigenvectors and generalized right eigenvectors of A associated with eigenvalues other than λ_α.

Remark 2. Proposition 4 implies the following facts, which will prove useful in the sequel. As above, let l denote a left eigenvector corresponding to the critical eigenvalue 0 (in Case (S)) or $i\omega_c$ (in Case (H)). Then $lw = 0$ if and only if $w \in E^s$. Moreover, $pw = 0$ for a row vector p if and only if $p \in \text{span}(l)$ (Case (S)), or $p \in \text{span}(\Re e\ l, \Im m\ l)$ (Case (H)). Finally, in Case H, we have $l\bar{r} = \bar{l}r = 0$.

Since the Jacobian matrix $L = Df(0)$ has part of its spectrum on the imaginary axis, it is not possible to choose a positive definite $\mathcal{P}$ for which $L^T\mathcal{P} + \mathcal{P}L$ is negative definite. However, one can ensure that the latter matrix is negative definite *on a subspace of* $I\!R^n$, while being only negative semidefinite on all of $I\!R^n$. A method for achieving this is given next.

Recall that r denotes the eigenvector of L corresponding to the critical eigenvalue (0 in Case S, $i\omega_c$ in Case (H)). Note that $r \in \mathbb{C}^n$ in Case H, and that $r \in I\!R^n$ in Case (S). Denote by E^s the subspace of $I\!R^n$ spanned by the eigenvectors (and generalized eigenvectors, if any) corresponding to the stable eigenvalues of L (in either Case (S) or Case (H)). We refer to E^s as the *stable subspace*

of $I\!R^n$.

The following proposition is useful in selecting the quadratic term $x^T\mathcal{P}x$ in the Liapunov function $\mathcal{V}$ under either hypothesis (S) or (H).

Proposition 5. (Liapunov Matrix Equation on Stable Subspace) Using the notation above, and under either hypothesis (S) or (H), there exists a family of real symmetric $n \times n$ matrices Π for which

$$\text{(i) } \Pi r = 0, \quad \text{(ii) } w^T \Pi w > 0, \text{ and } \quad \text{(iii) } w^T(L^T\Pi + \Pi L)w < 0 \tag{27}$$

for all $w \in E^s, w \neq 0$.

Proof. The proof may be carried out in two steps. In the first step, we exhibit a choice of coordinates for the state space $I\!R^n$ for which the existence of matrices Π is transparent. In the second step, we verify that the existence of a matrix Π satisfying (i)-(iii) in one coordinate system implies the existence of such a matrix for any choice of coordinates. *Step 1.* Suppose, then, that the state x of system (1) is expressed with respect to a coordinate basis $\{r^1, r^2, \ldots, r^n\}$ defined as follows. In Case S, take $r^1 := r$ and choose the remaining basis vectors $r^i \in I\!R^n$, $i = 2, \ldots, n$ such that $\text{span}\{r^2, \ldots, r^n\} = E^s$. Analogously, in Case H, choose $r^1 := \Re e(r)$, $r^2 := \Im m(r)$, and let $r^3, \ldots, r^n$ satisfy $\text{span}\{r^3, \ldots, r^n\} = E^s$. For such a coordinate basis, L has the block diagonal representation

$$L = \begin{pmatrix} \theta & 0 \\ 0 & L_s \end{pmatrix}. \tag{28}$$

Here, L_s is a real square *stable* matrix whose eigenvalues coincide with the stable eigenvalues of L, and θ is given by

$$\theta = \begin{cases} 0 & \text{in Case S,} \\ \begin{pmatrix} 0 & \omega_c \\ -\omega_c & 0 \end{pmatrix} & \text{in Case H.} \end{cases} \tag{29}$$

(Note that θ is a scalar for Case S, and is a 2×2 matrix for Case (H).) It is now straightforward to exhibit a matrix Π satisfying (i)-(iii). Consider the matrix

$$\Pi = \begin{pmatrix} 0 & 0 \\ 0 & \Pi_{22} \end{pmatrix}, \tag{30}$$

where Π_{22} is a real symmetric positive definite matrix of dimension $(n-1)$ in Case S, and dimension $(n-2)$ in Case H, for which the matrix

$$L_s{}^T \Pi_{22} + \Pi_{22} L_s \tag{31}$$

is negative definite. The existence of such a matrix Π_{22} is clear, since L_s is stable. Note that, for the present choice of coordinate basis, we have that in Case (S) the first component of any vector $w \in E^s$ is 0, and, in Case H, the first two components of w are 0. Also, the right eigenvector r is given, in Case (S) and Case H, by $r = (1, 0, \ldots, 0)^T$ and $r = (1, i, 0, \ldots, 0)^T$, respectively. The matrix Π of Eq. (30) is now easily verified to satisfy conditions (i)-(iii). *Step 2.* Next, we show that existence of a matrix Π satisfying (i)-(iii) in one set of coordinates implies existence of such a matrix for any set of coordinates. Let the coordinate change be determined by a nonsingular transformation matrix Σ, and indicate vectors in the transformed coordinates by a hat ($\widehat{\ }$):

$$\widehat{r} = \Sigma r, \quad \widehat{w} = \Sigma w. \tag{32}$$

In the new coordinates, L has the representation

$$\widehat{L} \;=\; \Sigma L \Sigma^{-1}. \tag{33}$$

It is straightforward to verify that the matrix

$$\widehat{\Pi} = (\Sigma^T)^{-1} \Pi \Sigma^{-1} \tag{34}$$

satisfies conditions (i)-(iii), where all quantities are taken in the new coordinates. (Note that the relationship between $\widehat{\Pi}$ and Π specified in Eq. (34) is not that of a similarity transformation.) □

As a corollary to Proposition 5, we have the following result.

Corollary 1. Let Π be a matrix satisfying (i)-(iii) of Proposition 5. Then, there is an $\alpha > 0$ such that, for each nonzero $w \in E^s$,

$$w^T (L^T \Pi + \Pi L) w < -\alpha |w|^2. \tag{35}$$

Proof. This follows easily using the proof of Proposition 5 and the standard fact that, for a negative definite real matrix Q, there is an $\alpha > 0$ such that $x^T Q x < -\alpha |x|^2$ for all $x \in I\!R^n$. In the present setting, the role of Q is played by ${L_s}^T \Pi_{22} + \Pi_{22} L_s$. □

In our construction of Liapunov functions for Case S, we shall employ matrices $\mathcal{P}$ of the form

$$\mathcal{P} = \Pi + l^T l \tag{36}$$

where Π is any real $n \times n$ matrix satisfying the conditions of Proposition 5. For Case H, the matrices $\mathcal{P}$ will be of the form

$$\mathcal{P} = \Pi + l^T \bar{l} + \bar{l}^{\,T} l. \tag{37}$$

It is not difficult to check that, both in Case (S) and Case H, the matrix $\mathcal{P}$ is positive definite, and the matrix $L^T\mathcal{P} + \mathcal{P}L$ is negative *semi*definite.

6 CONSTRUCTION OF LIAPUNOV FUNCTIONS IN THE CASE OF ONE ZERO EIGENVALUE

In this section, we construct a family of Liapunov function candidates for system (1) (equivalently, (13)) under hypothesis (S). The main task is to specify the matrix $\mathcal{P}$ and the cubic form $\mathcal{K}(x,x,x)$ appearing in the expression (3) for $\mathcal{V}(x)$. In the foregoing, we have constrained the matrix $\mathcal{P}$ to take the form $\mathcal{P} = l^T l + \Pi$ in Case S, where Π is any real symmetric $n \times n$ matrix satisfying the conditions of Proposition 5.

6.1 Conditions for $\dot{\mathcal{V}} < 0$

Using Proposition 1 (Jacobian of Homogeneous Functions), the time derivative of $\mathcal{V}(x)$ evaluated along trajectories of Eq. (13) can be written as

$$\begin{aligned} \dot{\mathcal{V}}(x) \;&=\; x^T(L^T\mathcal{P} + \mathcal{P}L)x \\ &\quad + 2Q^T(x,x)\mathcal{P}x + 3\mathcal{K}(x,x,Lx) \\ &\quad + 2C^T(x,x,x)\mathcal{P}x + 3\mathcal{K}(x,x,Q(x,x)) + \cdots \end{aligned} \tag{38}$$

Recall from Section 5 the notation E^s for the stable subspace of $I\!R^n$, i.e., the $(n-1)$-dimensional subspace spanned by the eigenvectors (and generalized eigenvectors, if any) corresponding to the stable eigenvalues of L. Using the representation $x = ar + w$ ($w \in E^s$) in Eq. (38); recalling that $\mathcal{P}$ has been chosen, by Eq. (36), such that $\mathcal{P} = l^T l + \Pi$ with $\Pi r = 0$; invoking the fact that a multilinear function is linear in each argument; and collecting terms on the right side of (38) of like order in $|(a,w)|$, we obtain a series expansion

$$\dot{\mathcal{V}}(x) := [\dot{\mathcal{V}}]_{(2)} + [\dot{\mathcal{V}}]_{(3)} + [\dot{\mathcal{V}}]_{(4)} + \cdots \tag{39}$$

where the integer subscripts denote the degree of the corresponding term in $|(a, w)|$, and the dots denote terms of fifth and higher order in $|(a, w)|$. Specifically, the terms appearing on the right side of Eq. (39) are given by

$$[\dot{\mathcal{V}}]_{(2)} = w^T(L^T\Pi + \Pi L)w, \tag{40}$$

$$\begin{aligned}[\dot{\mathcal{V}}]_{(3)} = {} & 2a^3 lQ(r,r) \\ & + 3a^2\mathcal{K}(r,r,Lw) + 4a^2 lQ(r,w) + 2a^2 Q^T(r,r)\Pi w \\ & + 2alQ(w,w) + 4aQ^T(r,w)\Pi w + 6a\mathcal{K}(r,w,Lw) \\ & + 2Q^T(w,w)\Pi w + 3\mathcal{K}(w,w,Lw),\end{aligned} \tag{41}$$

$$\begin{aligned}[\dot{\mathcal{V}}]_{(4)} = {} & a^4\{2lC(r,r,r) + 3\mathcal{K}(r,r,Q(r,r))\} \\ & + 2a^3\{3lC(r,r,w) + 3\mathcal{K}(r,w,Q(r,r)) + 3\mathcal{K}(r,r,Q(r,w))\} \\ & + 3a^2\{2lC(r,w,w) + \mathcal{K}(r,r,Q(w,w)) + 4\mathcal{K}(r,w,Q(r,w)) + \mathcal{K}(w,w,Q(r,r))\} \\ & + 2a\{lC(w,w,w) + 3\mathcal{K}(r,w,Q(w,w)) + 3\mathcal{K}(w,w,Q(r,w))\} \\ & + 2C^T(w,w,w)\Pi w + 3\mathcal{K}(w,w,Q(w,w)).\end{aligned} \tag{42}$$

Note that condition (iii) on Π (cf. Proposition 5) implies that $[\dot{\mathcal{V}}]_{(2)} < 0$ for $w \in E^s, w \neq 0$. This does not of course imply that $\dot{\mathcal{V}}$ is locally negative definite, only that it is locally negative definite on the subspace E^s.

Lemma 1, along with the foregoing computation of $\dot{\mathcal{V}}$, allow us to obtain the following preliminary statement concerning the local asymptotic stability of the origin of Eq. (1). Note that condition (S1) of the next proposition is a known necessary condition for stability for systems (1) possessing a simple zero eigenvalue (see for instance [2], [10]).

Proposition 6. Under hypothesis (S), the origin of Eq. (1) is locally asymptotically stable if there are a real symmetric $n \times n$ matrix Π satisfying (i)-(iii) of Proposition 5, and a symmetric real trilinear function $\mathcal{K}(x^1, x^2, x^3)$, for which the following three conditions hold:

(S1) $lQ(r,r) = 0$,

(S2) $3\mathcal{K}(r,r,Lw) + 4lQ(r,w) + 2Q^T(r,r)\Pi w = 0$ for all $w \in E^s$, and

(S3) $2lC(r,r,r) + 3\mathcal{K}(r,r,Q(r,r)) < 0$.

Proof. Let conditions (S1)-(S3) of the Proposition hold. It is straightforward to write an upper bound for $\dot{\mathcal{V}}(ar + w)$ in the form of a scalar bivariate Taylor series. To facilitate application of Lemma 1, we employ notation consistent with Eq. (19) of Lemma 1, and define variables $u := |w|$ and $v := |a|$. The proof proceeds in two steps. The first step consists of verifying that $a_{03} = 0$ and $a_{12} = 0$. In the second step, we ascertain that $a_{20} < 0$ and $a_{04} < 0$. *Step 1.* From Eq. (41), it is clear that the v^3-term in $\dot{\mathcal{V}}$ is $2lQ(r, r)v^3$, and this vanishes by virtue of condition (S1). Thus, the upper bound for $\dot{\mathcal{V}}$ will naturally be absent of a term $a_{03}v^3$. Using Eq. (41), it is apparent that the uv^2-term in the upper bound will vanish if $3\mathcal{K}(r, r, Lw) + 4lQ(r, w) + 2Q^T(r, r)\Pi w = 0$ for each $w \in E^s$. This latter condition is precisely (S2). *Step 2.* Eq. (40) and Corollary 1 imply that the quadratic terms in $\dot{\mathcal{V}}$ are bounded above by a function $a_{20}u^2$, where $a_{20} < 0$. Also, Eq. (42) (which gives the quartic terms in $\dot{\mathcal{V}}$) and assumption (S3) together imply that $a_{04} < 0$. □

6.2 Algorithm for Construction of $\mathcal{V}$ in Case (S)

For Proposition 6 to be useful in the explicit construction of Liapunov functions for Eq. (1), a method is needed facilitating the choice of a matrix Π and a trilinear function $\mathcal{K}$ for which, under an auxiliary condition guaranteeing local stability, (S2) and (S3) are satisfied. As it turns out, one can first choose any matrix Π for which conditions (i)-(iii) of Proposition 5 hold, and then proceed to construct compatible trilinear functions $\mathcal{K}$ satisfying (S2), (S3). We now proceed to construct a family of such trilinear functions, using the representation (17) in terms of the associated structural coefficients.

The general representation (17) for trilinear functions $\mathcal{K}(x^1, x^2, x^3)$ assumes a specific choice of basis for $I\!R^n$. Let $\{r^1, r^2, \ldots, r^n\}$ be a basis for $I\!R^n$, obtained by setting $r^1 := r$ and requiring that $r^i \in E^s$ for $i = 2, \ldots, n$. Let the associated dual basis (discussed in Section 2.3) be given by $\{l^1, l^2, \ldots, l^n\}$. Recall that the biorthonormality property (9) holds, i.e., that $l^i r^j = \delta_{ij}$, the Kronecker delta. Moreover, by Proposition 4 and the fact that the dual basis is unique, we have that $l^1 = l$, the left eigenvector of L associated with the eigenvalue zero (recall the normalization $lr = 1$).

By (17), the trilinear function $\mathcal{K}$ can be represented as follows:

$$\mathcal{K}(x^1, x^2, x^3) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \kappa_{ijk}(l^i x^1)(l^j x^2)(l^k x^3). \tag{43}$$

By (S1) and Proposition 4, we have that $Q(r,r) \in E^s$. Thus we are free to substitute $Q(r,r)$ for Lw in condition (S2), upon which we directly obtain

$$
\begin{aligned}
3\mathcal{K}(r,r,Q(r,r)) &= -4lQ(r,L^-Q(r,r)) - 2Q^T(r,r)\Pi L^- Q(r,r) \\
&= -4lQ(r,L^-Q(r,r)) - Q^T(r,r)((L^-)^T\Pi + \Pi L^-)Q(r,r).
\end{aligned} \tag{48}
$$

Here, L^- is as defined in Eq. (45). Substitution of (48) into (S3) yields the following condition equivalent to (S3):

$$
2lC(r,r,r) - 4lQ(r,L^-Q(r,r)) - Q^T(r,r)((L^-)^T\Pi + \Pi L^-)Q(r,r) < 0. \tag{49}
$$

Note that (S3) therefore places a condition only on the quadratic part of the Liapunov function candidate $\mathcal{V}$, as reflected by the appearance in (49) of the matrix Π.

Conditions (S1) and (S3) are akin to conditions that arise in the stability analysis of stationary bifurcation for parametrized embeddings of Eq. (1). Under hypothesis (S), such parametrized systems will generically exhibit a bifurcation in which a new equilibrium x_ϵ coexists with the origin for each small $|\epsilon|$. Here, ϵ is a (normalized) real amplitude parameter. The eigenvalue near zero of the bifurcated equilibrium x_ϵ is given by an expansion

$$
\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \beta_3\epsilon^3 + \cdots \tag{50}
$$

To guarantee asymptotic stability of the new equilibrium x_ϵ, one requires $\beta_1 = 0$ and $\beta_2 < 0$. In this context (cf. [2]), we have the "bifurcation formulae"

$$
\beta_1 = lQ(r,r), \tag{51}
$$

$$
\beta_2 = 2l\{C(r,r,r) - 2Q(r,L^-Q(r,r))\}. \tag{52}
$$

Note that condition (S1) is therefore identical to the bifurcation stability condition $\beta_1 = 0$. Similarly, (S3) (equivalently, Eq. (49)) is readily expressed in terms of the coefficient β_2. Denote by $\Delta_S(\Pi)$ the Π-dependent scalar

$$
\Delta_S(\Pi) := -Q^T(r,r)((L^-)^T\Pi + \Pi L^-)Q(r,r). \tag{53}
$$

Eq. (49) is now rewritten in terms of β_2 and Δ_S:

$$\beta_2 + \Delta_S(\Pi) < 0. \tag{54}$$

The following result is well known (see, e.g., [2], [9]).

Theorem 1. Let hypothesis (S) hold, and suppose that $\beta_1 = 0$ and $\beta_2 < 0$, where β_1 and β_2 are given by (51) and (52), respectively. Then the origin of Eq. (1) is locally asymptotically stable.

In our pursuit of Liapunov functions for (1), we have in fact rederived this result. Indeed, note that our condition (S1) requires that $\beta_1 = 0$, and, for a given system (1) for which $\beta_2 < 0$, the matrix Π can be chosen so as to ensure that (54) holds. (Recall that Π is any real symmetric matrix satisfying conditions (i)-(iii) of Proposition 5. These conditions are linear in Π.)

An observation of relevance here is that, for any choice of Π, the quantity $\Delta_S(\Pi)$ is nonnegative:

$$\Delta_S(\Pi) \geq 0. \tag{55}$$

Thus, $\Delta_S(\Pi)$ contributes adversely to satisfaction of Eq. (53).

We can now present the main result of this section, the construction of a family of Liapunov functions $\mathcal{V}(x)$ of the form (14) for Case (S) (one zero eigenvalue).

Theorem 2. Let hypothesis (S) hold, and suppose that $\beta_1 = 0$ and $\beta_2 < 0$, where β_1 and β_2 are given by (51) and (52), respectively. Then any function $\mathcal{V}(x)$ resulting from Algorithm $\mathcal{V}_S$ below is a Liapunov function for the equilibrium point 0 of Eq. (1).

Algorithm $\mathcal{V}_S$. (Construction of Liapunov functions $\mathcal{V}(x) = x^T\mathcal{P}x + \mathcal{K}(x,x,x)$ in Case (S))

Step 1. Compute l and r. Choose a basis $\{r^2, \ldots, r^n\}$ for E^s. Compute the dual basis $\{l^1, l^2, \ldots, l^n\}$ to the basis $\{r^1, r^2, \ldots, r^n\}$ for $I\!R^n$. Here, $r^1 := r$ and $l^1 = l$. Compute the coefficients β_1 and β_2 according to Eqs. (51) and (52), respectively. Check that $\beta_1 = 0$ and $\beta_2 < 0$.

Step 2. Choose any real symmetric $n \times n$ matrix Π satisfying, for all $w \in E^s$, $w \neq 0$: (i) $\Pi r = 0$, (ii) $w^T \Pi w > 0$, and (iii) $w^T(L^T\Pi + \Pi L)w < 0$, and for which

$$|\Delta_S(\Pi)| < |\beta_2|, \tag{56}$$

where $\Delta_S(\Pi)$ is as defined in Eq. (53).

Step 3. Set $\mathcal{P} = l^T l + \Pi$.

Step 4. Set the structural coefficients $\kappa_{i11}, i = 2, \ldots, n$ to

$$\kappa_{i11} = -\frac{2}{3}\{2lQ(r, L^- r^i) + Q^T(r,r)\Pi L^- r^i\}. \tag{57}$$

Step 5. Symmetry requires that κ_{ijk} be independent of permutations in the indices i, j, k. The structural coefficients in the representation

$$\mathcal{K}(x^1, x^2, x^3) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \kappa_{ijk}(l^i x^1)(l^j x^2)(l^k x^3) \tag{58}$$

which have not been specfied in Steps 1-4 are either determined by symmetry and Eq. (57), or can be assigned arbitrarily, subject only to the symmetry requirement.

7 CONSTRUCTION OF LIAPUNOV FUNCTIONS IN THE CASE OF A PAIR OF PURE IMAGINARY EIGENVALUES

In this section, Liapunov functions are constructed for the origin of Eq. (1) under hypothesis (H). The construction parallels that of the last section, while differing from it in several respects. For example, the Liapunov function candidates used in the previous section consist only of quadratic and cubic terms in the state. In this section, quartic terms also appear in the Liapunov function candidates (cf. Eq. (15)). If quartic terms were not included in the assumed form of the Liapunov function candidates, the construction would fail *generically* in Case (H). A second difference between the calculations of this and the preceding section concerns the adoption of complex notation. Although not essential, complex notation is both natural and convenient when considering local asymptotic stability of systems (1) under hypothesis (H).

7.1 Conditions for $\dot{\mathcal{V}} < 0$

Consider, then, the time-derivative of the Liapunov function candidate

$$\mathcal{V}(x) = x^T \mathcal{P} x + \mathcal{K}(x,x,x) + \mathcal{T}(x,x,x,x) \tag{59}$$

along trajectories of Eq. (1). Using Proposition 1, and the Taylor series representation (13) of $f(x)$, we find that this derivative is given by

$$\begin{aligned} \dot{\mathcal{V}}(x) &= x^T(L^T\mathcal{P} + \mathcal{P}L)x \\ &\quad + 2Q^T(x,x)\mathcal{P}x + 3\mathcal{K}(x,x,Lx) \\ &\quad + 2C^T(x,x,x)\mathcal{P}x + 3\mathcal{K}(x,x,Q(x,x)) + 4\mathcal{T}(x,x,x,Lx) + \cdots \end{aligned} \tag{60}$$

In Section 5, it was noted that any vector $x \in I\!R^n$ has a unique representation

$$x = ar + \bar{a}\bar{r} + w. \tag{61}$$

Here, r is the right eigenvector of L associated with the eigenvalue $i\omega_c$ (as specified in Section 1), $\bar{r}$ is the complex conjugate of r and as such is a right eigenvector of L associated with the eigenvalue $-i\omega_c$, a is a complex scalar, and $w \in E^s$.

The first step in our procedure for obtaining conditions for local negative definiteness of $\dot{\mathcal{V}}(x)$ is to substitute in Eq. (60) the representation (61) for x and the representation (37) for the matrix $\mathcal{P}$, and group terms according to their order in $|(a,\bar{a},w)|$. (Recall that (37) states that, in Case II, $\mathcal{P}$ is chosen from among matrices of the form $\mathcal{P} = \Pi + l^T\bar{l} + \bar{l}^{\,T}l$, where Π satisfies conditions (i)-(iii) of Proposition 5.) Using Proposition 4 and Remark 2 (orthogonality of left and right eigenvectors), and after a considerable amount of algebra and reordering of terms, we can obtain explicit formulae for the quadratic, cubic and quartic terms in $|(a,\bar{a},w)|$ (these are $[\dot{\mathcal{V}}]_{(2)}$, $[\dot{\mathcal{V}}]_{(3)}$, and $[\dot{\mathcal{V}}]_{(4)}$, respectively) in the expansion

$$\dot{\mathcal{V}}(x) := [\dot{\mathcal{V}}]_{(2)} + [\dot{\mathcal{V}}]_{(3)} + [\dot{\mathcal{V}}]_{(4)} + \cdots \tag{62}$$

of $\dot{\mathcal{V}}(x)$. (This is in analogy with Eqs. (39)-(42) of the preceding section.) For example, one can check that $[\dot{\mathcal{V}}]_{(2)}$, $[\dot{\mathcal{V}}]_{(3)}$ are given by

$$[\dot{\mathcal{V}}]_{(2)} = w^T(L^T\Pi + \Pi L)w, \tag{63}$$

$$
\begin{aligned}
[\dot{\mathcal{V}}]_{(3)} \;=\;& a^3\{2\bar{l}Q(r,r) + 3i\omega_c\mathcal{K}(r,r,r)\} \\
&+ \bar{a}^3\{2lQ(\bar{r},\bar{r}) - 3i\omega_c\mathcal{K}(\bar{r},\bar{r},\bar{r})\} \\
&+ a^2\bar{a}\{2lQ(r,r) + 4\bar{l}Q(r,\bar{r}) + 3i\omega_c\mathcal{K}(r,r,\bar{r})\} \\
&+ \bar{a}^2 a\{2\bar{l}Q(\bar{r},\bar{r}) + 4lQ(r,\bar{r}) - 3i\omega_c\mathcal{K}(r,\bar{r},\bar{r})\} \\
&+ a^2\{2Q^T(r,r)\Pi w + 4\bar{l}Q(r,w) + 3\mathcal{K}(r,r,Lw) + 6i\omega_c\mathcal{K}(r,r,w)\} \\
&+ \bar{a}^2\{2Q^T(\bar{r},\bar{r})\Pi w + 4lQ(\bar{r},w) + 3\mathcal{K}(\bar{r},\bar{r},Lw) - 6i\omega_c\mathcal{K}(\bar{r},\bar{r},w)\} \\
&+ a\bar{a}\{4Q^T(r,\bar{r})\Pi w + 4\bar{l}Q(\bar{r},w) + 4lQ(r,w) + 6\mathcal{K}(r,\bar{r},Lw)\} \\
&+ a\{2\bar{l}Q(w,w) + 6\mathcal{K}(r,w,Lw) + 3i\omega_c\mathcal{K}(r,w,w)\} \\
&+ \bar{a}\{2lQ(w,w) + 6\mathcal{K}(\bar{r},w,Lw) - 3i\omega_c\mathcal{K}(\bar{r},w,w)\} \\
&+ 3\mathcal{K}(w,w,Lw). \qquad (64)
\end{aligned}
$$

Note that both $[\dot{\mathcal{V}}]_{(2)}$ and $[\dot{\mathcal{V}}]_{(3)}$ as given above are real-valued, as expected. There is no need to give the full expression for $[\dot{\mathcal{V}}]_{(4)}$ here. Instead, we proceed directly to the statement of the following preliminary result in the construction of Liapunov functions under hypothesis (H). In the proof of this result, which is analogous to Proposition 6 in the preceding section, values of certain pertinent terms appearing in the expansion of $[\dot{\mathcal{V}}]_{(4)}$ will be given. All the terms in the expansion of $[\dot{\mathcal{V}}]_{(4)}$ may be obtained readily, to result in an expression analogous to (64).

Proposition 7. Let hypothesis (H) hold. Suppose that, for some real matrix Π satisfying the conditions of Proposition 5, symmetric real trilinear function $\mathcal{K}(x^1, x^2, x^3)$, and symmetric real tetralinear function $\mathcal{T}(x^1, x^2, x^3, x^4)$, the following seven conditions hold:

(H1) $2\bar{l}Q(r,r) + 3i\omega_c\mathcal{K}(r,r,r) = 0$,

(H2) $2lQ(r,r) + 4\bar{l}Q(r,\bar{r}) + 3i\omega_c\mathcal{K}(r,r,\bar{r}) = 0$,

(H3) $2Q^T(r,r)\Pi w + 4\bar{l}Q(r,w) + 3\mathcal{K}(r,r,(L+2i\omega_c I)w) = 0$, for all $w \in E^s$,

(H4) $2Q^T(r,\bar{r})\Pi w + 2lQ(r,w) + 2\bar{l}Q(\bar{r},w) + 3\mathcal{K}(r,\bar{r},Lw) = 0$, for all $w \in E^s$,

(H5) $2\bar{l}C(r,r,r) + 3\mathcal{K}(r,r,Q(r,r)) + 4i\omega_c\mathcal{T}(r,r,r,r) = 0$,

(H6) $lC(r,r,r) + 3\bar{l}C(r,r,\bar{r}) + +3\mathcal{K}(r,r,Q(r,\bar{r})) + 3\mathcal{K}(r,\bar{r},Q(r,r)) + 4i\omega_c\mathcal{T}(r,r,r,\bar{r}) = 0$,

(H7) $\Re e\{2lC(r,r,\bar{r}) + \mathcal{K}(r,r,Q(\bar{r},\bar{r}))\} + 2\mathcal{K}(r,\bar{r},Q(r,\bar{r})) < 0$.

Then the origin of Eq. (1) is locally asymptotically stable.

Proof. The proof consists of a judicious application of Lemma 1, which gives general sufficient conditions for local negative definiteness of a class of bivariate functions. Identify the scalar variables u and v in Lemma 1 as $u := |w|$, $v := |a| = |\bar{a}|$. The bivariate function $\delta(u,v)$ of Lemma 1 is taken to be a local upper bound on $\dot{\mathcal{V}}(x)$. We show that conditions (H1)-(H7) are sufficient for there to exist such an upper bound $\delta(u,v)$ satisfying the hypotheses of Lemma 1. To ensure absence of the v^3-term in $\delta(u,v)$, i.e., that $a_{03} = 0$, we require the coefficients of a^3, $\bar{a}^3$, $a^2\bar{a}$ and $\bar{a}^2 a$ in Eq. (64) to vanish. The coefficient of a^3 is precisely the expression on the left side of (H1). (Note that the $\bar{a}^3$-term in (64) is the complex conjugate of the a^3-term, and thus also vanishes when (H1) is in force; analogous comments apply below.) Similarly, the expression on the left side of (H2) is simply the coefficient of $a^2\bar{a}$ in Eq. (64). Thus, (H1) and (H2) *combined* ensure that $a_{03} = 0$. To ensure absence of the uv^2-term in $\delta(u,v)$, i.e., that $a_{12} = 0$, we require the "linear-in-w" coefficients of a^2 and $a\bar{a}$ in Eq. (64) to vanish for each $w \in E^s$. (The coefficient of $\bar{a}^2$, being the conjugate of that of a^2, will then vanish automatically.) Inspection of Eq. (64) reveals that this is equivalent to conditions (H3) and (H4) above. It remains to show that $a_{20} < 0$ and $a_{40} < 0$ (in the notation of Lemma 1). That $a_{20} < 0$ follows immediately from Eq. (63) and Corollary 1. Conditions ensuring that $a_{40} < 0$ can only result from examination of the quartic terms in $\dot{\mathcal{V}}$. However, quartic terms in $[\dot{\mathcal{V}}]_{(4)}$ which involve w are irrelevant to this requirement. The needed coefficients can be obtained readily by substituting Eq. (61) for x in the formula

$$[\dot{\mathcal{V}}]_{(4)}(x) = 2C^T(x,x,x)\mathcal{P}x + 3\mathcal{K}(x,x,Q(x,x)) + 4\mathcal{T}(x,x,x,Lx), \tag{65}$$

using the orthogonality of left and right eigenvectors (cf. Proposition 4 and Remark 2), and using the fact that a multilinear form is linear in each argument. The expansion of $[\dot{\mathcal{V}}]_{(4)}(ar + \bar{a}\bar{r} + w)$ is seen to contain five terms which are quartic in $(a,\bar{a})$: an a^4-term, an $a^3\bar{a}$-term, the conjugates of these two, and an $a^2\bar{a}^2$-term. Of these five, only the latter is sign-definite: $a^2\bar{a}^2 = |a|^4$. Thus, we require the coefficient of $a^2\bar{a}^2$ in the expansion of $[\dot{\mathcal{V}}]_{(4)}(ar + \bar{a}\bar{r} + w)$ to be negative, and the coefficients of a^4 and $a^3\bar{a}$ to vanish. It is readily verified that the left side of (H5) is the coefficient of a^4, and the left side of (H6) is half the coefficient of $a^3\bar{a}$. Finally, it is straightforward to check that the left side of (H7) is one-sixth the coefficient of $a^2\bar{a}^2$ in the expansion of $[\dot{\mathcal{V}}]_{(4)}(ar + \bar{a}\bar{r} + w)$.

Thus, conditions (H5)-(H7) together give the desired negativity of a_{04}. $\square$

7.2 Algorithm for Construction of $\mathcal{V}$ in Case (H)

Conditions (H1)-(H7) may be solved for a trilinear function $\mathcal{K}(x^1, x^2, x^3)$ and a tetralinear function $\mathcal{T}(x^1, x^2, x^3, x^4)$, under an appropriate auxiliary condition ensuring stability of the origin and for a given matrix Π satisfying the conditions of Proposition 5. The procedure is much the same as was carried out in the preceding section, where the coordinate representation of a trilinear form $\mathcal{K}(x^1, x^2, x^3)$ was employed to solve (S1)-(S3) for the structural coefficients κ_{ijk} of $\mathcal{K}$. Due to this similarity, only a summary of the main steps in the derivation is deemed necessary here, with the result for the Liapunov functions we obtain summarized below in Algorithm $\mathcal{V}_H$.

For convenience, we continue to employ complex notation, and choose a coordinate basis $\{r^1, r^2, \ldots, r^n\}$ for $I\!R^n$ in which $r^1 := r$, $r^2 := \bar{r}$, and $r^3, \ldots, r^n$ lie in $E^s \subset I\!R^n$. To this basis there corresponds a unique dual basis of row vectors $\{l^1, l^2, \ldots, l^n\}$, where $l^1 := l$ and $l^2 := \bar{l}$. The trilinear and tetralinear functions $\mathcal{K}(x^1, x^2, x^3)$ and $\mathcal{T}(x^1, x^2, x^3, x^4)$ are then expressed in the coordinate representations (17) and (18), respectively. We seek the minimum set of specifications on the associated structural coefficients κ_{ijk} and τ_{ijkp}, respectively, under which conditions (H1)-(H7) above hold.

Conditions (H1) and (H2) are interpreted in this framework simply as assigning values to the structural coefficients κ_{111} and κ_{112}, respectively. Next consider (H3). Since E^s is invariant under L, $w \in E^s$ implies that $(L + 2i\omega_c I)w$ lies in the complexification of E^s (also referred to as E^s below). Thus, we can define a vector $\tilde{w} := (L + 2i\omega_c I)w$, noting that the matrix inverse in the equation $w = (L + 2i\omega_c I)^{-1}\tilde{w}$ exists by hypothesis (H). Interpreting (H3) as a requirement on each *basis vector* $r^3, \ldots, r^n$ of E^s, we find that (H3) amounts to a specification of the structural coefficients κ_{i11}, $i = 3, 4, \ldots, n$. Similarly, (H4) amounts to a specification of the structural coefficients κ_{i12}, $i = 3, 4, \ldots, n$.

Since each of the structural coefficients κ_{i11}, $i = 1, 2, \ldots, n$ is fixed (by one of (H1)-(H3)), $\mathcal{K}(r, r, x)$ is fixed *for any* $x \in \mathbb{C}^n$. The coefficients κ_{i12}, $i = 1, 2, \ldots, n$ are also fixed: κ_{112} is fixed by (H2), κ_{i12}, $i = 3, 4, \ldots, n$ are fixed by (H4), and we also have $\kappa_{212} = \overline{\kappa_{112}}$ by applying Proposition 3 to (H2). Thus, $\mathcal{K}(r, \bar{r}, x)$ is also fixed for any $x \in \mathbb{C}^n$. By these remarks, it follows that the terms $\mathcal{K}(r, r, Q(r, r))$, $\mathcal{K}(r, r, Q(r, \bar{r}))$, $\mathcal{K}(r, \bar{r}, Q(r, r))$ appearing in (H5) and (H6) are determined by

(H1)-(H4). Their values may be found by expressing $Q(r,r)$ and $Q(r,\bar{r})$ as linear combinations of the basis vectors r^i, $i = 1,\ldots,n$, and then employing the coordinate representation of $\mathcal{K}$. Thus, (H5) and (H6) serve to assign the values of $\mathcal{T}(r,r,r,r)$ ($= \tau_{1111}$) and $\mathcal{T}(r,r,r,\bar{r})$ ($= \tau_{1112}$). The importance of including the quartic term $\mathcal{T}$ in the Liapunov function candidate now becomes clear: with $\mathcal{T} = 0$, (H5) and (H6) become constraints on the system which do not constitute necessary conditions for stability. However, with inclusion of a quartic term $\mathcal{T}$, (H5) and (H6) are quite easily satisfied.

By the remarks above, it follows that the quantity appearing on the left side of (H7) is completely specified by the system dynamics and the matrix Π. Next, we sketch the derivation of an explicit reformulation of the left side of (H7) in terms of system (1) and Π. A stability coefficient which arises in the study of Hopf bifurcation for parametrized versions of (1) under hypothesis (H) will appear in the reformulation. The value of this coefficient, which we denote as β_2, is recalled next. Note that this coefficient β_2 is distinct from the coefficient of the same name appearing in Section 6. In the present context, β_2 relates to an (even in ϵ) expansion $\beta(\epsilon) = \beta_2\epsilon^2 + \beta_4\epsilon^4 + \cdots$ of the *Floquet exponent* near zero of bifurcated periodic solutions of parametrized embeddings of Eq. (1). (Compare with Eq. (50) for the analogous *eigenvalue* expansion in Case (S).)

Define vectors ξ and η by

$$\xi \;:=\; -\frac{1}{2}L^{-1}Q(r,\bar{r}), \tag{66}$$

$$\eta \;:=\; \frac{1}{2}(2i\omega_c I - L)^{-1}Q(r,r). \tag{67}$$

(In [9] and [1], ξ and η are denoted as a and b, respectively.) Then β_2 is given by the "bifurcation formula" [9], [1]

$$\beta_2 := 2\Re e\{2lQ(r,\xi) + lQ(\bar{r},\eta) + \frac{3}{4}lC(r,r,\bar{r})\}. \tag{68}$$

Conditions (H1)-(H4) can be used to replace (H7) with an equivalent condition stated explicitly in terms of the Taylor expansion (13) of system (1). Consider the two terms $\mathcal{K}(r,r,Q(\bar{r},\bar{r}))$ and $\mathcal{K}(r,\bar{r},Q(r,\bar{r}))$ appearing in (H7). The former quantity is of the form $\mathcal{K}(r,r,x)$, which occurs in the statements of conditions (H1) (with $x = r$), (H2) (with $x = \bar{r}$) and (H3) (with $x \in E^s$). Thus, we resolve the vector x ($= Q(\bar{r},\bar{r})$) into its components lying in the subspace E^s and along the r- and $\bar{r}$-directions, and then apply (H1)-(H3). The r-component of any vector x is given by $(lx)r$ (recall the normalization $lr = 1$), the $\bar{r}$-component is $(\bar{l}x)\bar{r}$, and the E^s-component is the remainder

$$x^s := x - (lx)r - (\bar{l}x)\bar{r}. \tag{69}$$

Using (H1)-(H3), we have that, for any x,

$$\begin{aligned}\mathcal{K}(r,r,x) &= -\frac{2}{3i\omega_c}(lx)\bar{l}Q(r,r) - \frac{2}{3i\omega_c}(\bar{l}x)(lQ(r,r) + 2\bar{l}Q(r,\bar{r})) \\ &\quad - \frac{1}{3}\{2Q^T(r,r)\Pi(L+2i\omega_c I)^{-1}x^s + 4\bar{l}Q(r,(L+2i\omega_c I)^{-1}x^s\}.\end{aligned} \tag{70}$$

Since r is an eigenvector of L, it is also an eigenvector of $(L+2i\omega_c I)^{-1}$ and of $(L-2i\omega_c I)^{-1}$. Using this fact, and the fact that $\Pi r = 0$, this expression may be expanded and simplified. Letting $x = Q(\bar{r},\bar{r})$, the resulting expression is

$$\begin{aligned}\mathcal{K}(r,r,Q(\bar{r},\bar{r})) &= \frac{8}{3}\bar{l}Q(r,\bar{\eta}) + \frac{4}{3}Q^T(r,r)\Pi\bar{\eta} \\ &\quad - \frac{2}{9i\omega_c}(lQ(\bar{r},\bar{r}))(\bar{l}Q(r,r)) - \frac{2}{3i\omega_c}(\bar{l}Q(\bar{r},\bar{r}))(lQ(r,r)).\end{aligned} \tag{71}$$

Thus,

$$\Re e(\mathcal{K}(r,r,x)) = \Re e\{\frac{8}{3}\bar{l}Q(r,\bar{\eta}) + \frac{4}{3}Q^T(r,r)\Pi\bar{\eta}\}. \tag{72}$$

The following formula for $\mathcal{K}(r,\bar{r},Q(r,\bar{r}))$ is obtained in a similar fashion. The fact that r and $\bar{r}$ are eigenvectors of L^{-1} is employed in the computation.

$$\mathcal{K}(r,\bar{r},Q(r,\bar{r})) = \frac{8}{3}\Re e\{lQ(r,\xi) + \frac{4}{3}Q^T(r,\bar{r})\Pi\xi\}. \tag{73}$$

Condition (H7) may now be rewritten explicitly as

$$\beta_2 + \Delta_H(\Pi) < 0, \tag{74}$$

where β_2 is as defined above, and where Δ_H is given by the real number

$$\begin{aligned}\Delta_H(\Pi) &:= Q(r,\bar{r})^T((L^{-1})^T\Pi + \Pi L^{-1})Q(r,\bar{r}) \\ &\quad - \frac{1}{4}Q^T(r,r)\{\Pi(L+2i\omega_c I)^{-1} + ((L-2i\omega_c I)^{-1})^T\Pi\}Q(\bar{r},\bar{r}).\end{aligned} \tag{75}$$

It is not difficult to ascertain that Δ_H is nonnegative, although this is not an essential consideration. (One proof of this uses the fact that formula (75) is a special case of the sum of a Hermitian form and a quadratic form.)

We have just rederived the following known criterion for asymptotic stability in Case (H).

Theorem 3. Let hypothesis (H) hold, and suppose that $\beta_2 < 0$, where β_2 is given by Eq. (68). Then the origin of (1) is locally asymptotically stable.

Regarding the structural coefficients κ_{ijk} and τ_{ijkp} that have not been specified explicitly in the foregoing analysis, only two constraints remain: The first, the *symmetry* requirement, entails that the value of a coefficient is independent of the order of subscript indices. The second constraint is that the function $\mathcal{V}(x)$ must be *real-valued* for $x \in I\!R^n$. Using Proposition 3, it is found that this latter requirement is equivalent to what might be called a *conjugate symmetry* relationship among the structural coefficients, the exact nature of which is specified in the next Corollary to Proposition 3.

Corollary 2. (Conjugate Symmetry of Structural Coefficients in Case (H)) Denote, for any positive integer i, the quantity $[i]$ (the "complement of i")

$$[i] := \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{if } i = 2, \\ i & \text{otherwise} \end{cases} \tag{76}$$

Then, $\mathcal{V}(x)$ is real-valued for each $x \in I\!R^n$ if and only if the structural coefficients κ_{ijk} and τ_{ijkp} satisfy the following relationship:

$$\overline{\kappa}_{ijk} = \kappa_{[i][j][k]}, \tag{77}$$

$$\overline{\tau}_{ijkp} = \tau_{[i][j][k][p]}. \tag{78}$$

Proof. Follows immediately from Proposition 3. □

The foregoing construction of a family of Liapunov functions $\mathcal{V}(x)$ of the form (59) for the case in which $Df(0)$ possesses a pair of pure imaginary eigenvalues is summarized in the next result and algorithm.

Theorem 4. Let hypothesis (H) hold, and suppose that $\beta_2 < 0$, where β_2 is given by Eq. (68). Then any function $\mathcal{V}(x)$ resulting from Algorithm $\mathcal{V}_H$ below is a Liapunov function for the equilibrium point 0 of Eq. (1).

Algorithm $\mathcal{V}_H$. (Construction of Liapunov functions $\mathcal{V}(x) = x^T \mathcal{P} x + \mathcal{K}(x, x, x) + \mathcal{T}(x, x, x, x)$ in Case (H))

Step 1. Compute l and r and choose a basis $\{r^1, r^2, \ldots, r^n\}$ for $I\!R^n$ with $r^1 := r$, $r^2 := \bar{r}$, and for which $\{r^3, \ldots, r^n\}$ is a basis for E^s. Calculate the row vectors $\{l^3, \ldots, l^n\}$ of the associated dual basis. Check that $\beta_2 < 0$, where β_2 is as defined in (68).

Step 2. Pick any Π satisfying (i) $\Pi r = \Pi \bar{r} = 0$, (ii) $w^T \Pi w > 0$, and (iii) $w^T(L^T \Pi + \Pi L)w < 0$ for all $w \in E^s, w \neq 0$, and such that $\Delta_H(\Pi) < |\beta_2|$, where $\Delta_H(\Pi)$ is given by Eq. (75).

Step 3. Set $\mathcal{P} = \Pi + l^T \bar{l} + \bar{l}^{\,T} l$.

Step 4. Set $\mathcal{K}(x^1, x^2, x^3) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \kappa_{ijk}(l^i x^1)(l^j x^2)(l^k x^3)$, and

$$\kappa_{111} = -\frac{1}{3i\omega_c}\bar{l}Q(r, r), \tag{79}$$

$$\kappa_{112} = -\frac{1}{3i\omega_c}\{4\bar{l}Q(r, \bar{r}) + 2lQ(r, r)\}, \tag{80}$$

$$\kappa_{i11} = -\frac{1}{3}\{4\bar{l}Q(r, (2i\omega_c I + L)^{-1} r^i) + 2Q^T(r, r)\Pi(L + 2i\omega_c I)^{-1} r^i\}, \quad i = 3, \ldots, n, \tag{81}$$

$$\kappa_{i12} = -\frac{2}{3}\{\bar{l}Q(\bar{r}, L^{-1} r^i) + lQ(r, L^{-1} r^i) + Q^T(r, \bar{r})\Pi L^{-1} r^i\}, \quad i = 3, \ldots, n. \tag{82}$$

Step 5. Set $\mathcal{T}(x^1, x^2, x^3, x^4) = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \sum_{p=1}^n \tau_{ijkp}(l^i x^1)(l^j x^2)(l^k x^3)(l^p x^4)$. Here, τ_{1111} ($= \mathcal{T}(r, r, r, r)$) and τ_{1112} ($= \mathcal{T}(r, r, r, \bar{r})$) are selected according to (H5) and (H6), respectively.

Step 6. All structural coefficients κ_{ijk} and τ_{ijkp} which have not been specified in Steps 1-5 are assigned arbitrarily, modulo the symmetry requirement and the conjugate symmetry requirement (Eqs. (77), (78)).

8 CONCLUDING REMARKS

Liapunov functions for nonlinear systems with either of the two simplest critical cases have been explicitly constructed. For the case in which the sytem linearization possesses a simple zero eigenvalue, generically the Liapunov functions need contain only quadratic and cubic terms in the state. However, when a complex conjugate pair of simple, pure imaginary eigenvalues are present, the Liapunov functions contain quartic terms, in addition to the quadratic and cubic terms. These Liapunov functions were shown to predict local asymptotic stability precisely when certain known sufficient conditions from bifurcation analysis are satisfied. We have obtained parametrized "families" of Liapunov functions for the studied critical cases, in the same sense that the Liapunov matrix equation yields an infinite set of quadratic Liapunov functions for asymptotically stable linear time-invariant systems. The Liapunov functions are computed directly in terms of the Taylor series expansion of the vector field $f(x)$, and are thus amenable to symbolic computer coding. The use of these Liapunov functions in the design of feedback control laws for critical nonlinear systems is a topic for future investigation.

APPENDIX A. SOLUTION OF LINEAR ALGEBRAIC EQUATIONS WITH SINGULAR COEFFICIENT MATRIX

Consider the system of linear equations

$$Ax = b \tag{A.1}$$

where A is a real $n \times n$ matrix and $b \in I\!R^n$. Suppose that A has a simple zero eigenvalue. Let r and l denote right (column) and left (row) eigenvectors of A, respectively, corresponding to the zero eigenvalue, and require that these be chosen to satisfy $lr = 1$. Under these conditions, the Fredholm Alternative asserts that (A.1) has a solution if and only if $lb = 0$. Moreover, the Fredholm Alternative also implies that, if (A.1) has a solution x^0, then the totality of solutions is given by the one-parameter family $x = x^0 + \alpha r$ where $\alpha \in I\!R$ is arbitrary. The solution is rendered unique upon imposing a normalization condition which specifies the value of lx.

Introduce subspaces $E^c, E^s \subset I\!R^n$ as follows: E^c is the one-dimensional subspace

$$E^c := \text{span}\{r\}, \tag{A.2}$$

and E^s is the $(n-1)$-dimensional subspace

$$E^s := \{x \in I\!R^n | \; lx = 0\}. \tag{A.3}$$

From the foregoing, we have in particular that if $lb = 0$ then the system $Ax = b$, $lx = 0$ has a unique solution. *Equivalently,* (A.1) ***has a unique solution in*** E^s ***for any vector*** $b \in E^s$. This proves that the *restriction* [8, p. 199] $A|_{E^s}$ of the linear map A to E^s defines an invertible (one-to-one and onto) map. In the next result, we exhibit the unique solution which lies in $E^s$of the system $Ax = b$, $lx = 0$. The proof is elementary [2].

Proposition A.1 The unique solution of $Ax = b$, $lx = 0$ given that $lb = 0$ is

$$x = (A^T A + l^T l)^{-1} A^T b. \tag{A.4}$$

This result motivates the following introduction of notation:

$$A^- := (A^T A + l^T l)^{-1} A^T. \tag{A.5}$$

Thus, the inverse of the restricted map $A|_{E^s}$ exists and is given by

$$(A|_{E^s})^{-1} = A^-. \tag{A.6}$$

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